1) Tractrix: $\alpha : (0, \pi) \to \mathbb{R}^2$

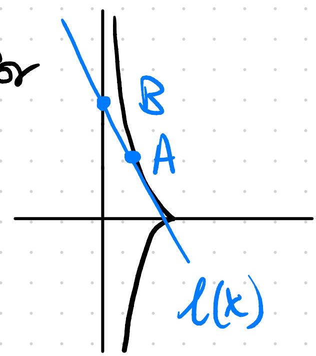
$$\alpha(t) = (\sin t, \cos t + \log \tan \tfrac{t}{2})$$

a) $\alpha'(t) = (\cos t, -\sin t + \csc t)$

at $t = \frac{\pi}{2}$, $\cos(\frac{\pi}{2}) = 0$

$$-\sin(\tfrac{\pi}{2}) + \csc(\tfrac{\pi}{2}) = 0.$$

So α is not regular at $t = \frac{\pi}{2}$. For any other $t \in (0, \pi)$, $\cos t \neq 0$, so α is regular everywhere else.



b) Direction of tangent line is given by expression for $\alpha'(t)$ above.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t + \csc t}{\cos t}$$

So equation of tangent line is:

$$\frac{y - (\cos t + \log \tan t/2)}{x - \sin t} = \frac{-\sin t + \csc t}{\cos t}$$

at y-axis, $x = 0$.

$$\Rightarrow y - \cos t - \log \tan t/2 = \frac{\sin^2 t - \sin t \csc t}{\cos t} = -\cos t$$

$$\Rightarrow y = \log \tan t/2.$$

So the point $A = \alpha(t) = (\sin t, \cos t + \log\tan t/2)$

$B = (0, \log\tan t/2)$.

So length of line segment $\overline{AB}$:

$$\overline{AB} = \sqrt{(\sin t - 0)^2 + (\cos t + \log\tan t/2 - \log\tan t/2)^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t} = 1$$

2) Suppose α is a cylindrical helix. WTS $\kappa/\tau = \text{const}$.

We have $\langle T, u \rangle = \cos\theta_0$. Then differentiating, we have

$$0 = \langle T', u \rangle = \langle \kappa N, u \rangle.$$

$\kappa > 0$, so $\langle N, u \rangle = 0$, i.e. u is perpendicular to N.

Then writing u in the basis $\{T, N, B\}$, we have

$$u = \cos\theta_0 T + \sin\theta_0 B.$$

WLOG u is a unit vector, then B-component of u is $\sqrt{1-\cos^2\theta_0} = \sqrt{\sin^2\theta_0} = \sin\theta_0$.

Then differentiating this gives

$$(\kappa\cos\theta_0 - \tau\sin\theta_0)N = 0 \Rightarrow \kappa\cos\theta_0 = \tau\sin\theta_0$$

using sign convention $B' = -\tau N$. Can also use opposite sign convention.

$$\Rightarrow \frac{\kappa}{\tau} = \tan\theta_0, \text{ a constant.}$$

$= -\tan\theta_0$ using other sign convention.

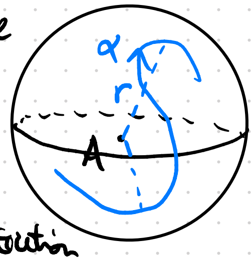
Now suppose $\frac{\kappa}{\tau}$ is a constant. Then writing

$\frac{\kappa}{\tau} = \tan\theta_0$ for some θ_0. Then defining $u = \cos\theta_0 T + \sin\theta_0 B$, we can check

$$\langle T, u \rangle = \langle T, \cos\theta_0 T + \sin\theta_0 B \rangle = \cos\theta_0 \text{ a constant,}$$

and that u is constant.

3) Necessity: First suppose α lies on a sphere. Then there is some r s.t. $|\alpha(s)|^2 = r^2$ for all s.



Following the hint, we differentiate $|\alpha(s)|$:

$$0 = \frac{d}{ds}|\alpha(s)|^2 = 2\alpha' \cdot \alpha$$ where $'$ denotes differentiation wrt s.

$\Rightarrow \alpha' \cdot \alpha = 0$. (i.e. $\alpha \cdot T = 0$).

Differentiating again, we have

$$0 = \alpha' \cdot \alpha' + \alpha'' \cdot \alpha \overset{\alpha'' = \kappa N,\ \alpha' \cdot \alpha' = 1}{\Longrightarrow} \kappa N \cdot \alpha = -1 \Rightarrow \alpha \cdot N = \frac{-1}{\kappa}.$$

and again, we have

$$0 = 3\alpha'' \cdot \alpha' + \alpha''' \cdot \alpha$$

So we have $\alpha'' \cdot \alpha' = 0$. So we have

$$\alpha''' \cdot \alpha = 0.$$

$\alpha''' = (\kappa N)'$ so we have $(\kappa' N + \kappa N') \cdot \alpha = 0$

$(\kappa' N + \kappa(-\kappa T + \tau B)) \cdot \alpha = 0$ Using sign convention $B' = -\tau N$.

$\Rightarrow \kappa' N \cdot \alpha + \kappa \tau B \cdot \alpha = 0$. And using $\alpha \cdot N = \frac{-1}{\kappa}$,

$\Rightarrow -\frac{\kappa'}{\kappa} + \kappa\tau\, \alpha \cdot B = 0. \Rightarrow \alpha \cdot B = \frac{\kappa'}{\kappa\tau}$

Writing α in the Frenet frame

$$\alpha = (\cancel{\alpha \cdot T})^{=0} T + (\alpha \cdot N)N + (\alpha \cdot B)B$$

we have

$|\alpha(s)|^2 = r^2 = (\alpha \cdot N)^2 + (\alpha \cdot B)^2$ by Pythagorean Thm.

So $\alpha = \frac{-1}{\kappa} N + \frac{\kappa'}{\kappa^2 \tau} B = -\rho N - \rho' \lambda B$

and we have

$$r^2 = \left(\frac{-1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2 \tau}\right)^2 = \left(\frac{1}{\kappa}\right)^2 + \left(\frac{-\kappa'}{\kappa^2}\right)^2 \left(\frac{1}{\tau}\right)^2$$

$$= \rho^2 + (\rho')^2 \lambda^2 \text{ is a constant.}$$

Sufficiency: Defining $\beta(s) = \alpha + \rho N + \rho' \lambda B$

alternatively $\beta(s) = \alpha + \rho N - \rho' \lambda B$ using the sign convention $B' = \tau N$.

We'll show β is a constant:

$$\beta' = \alpha' + \rho' N + \rho N' + (\rho' \lambda)' B + (\rho' \lambda) B'$$

$$= \left(\rho \tau + (\rho' \lambda)'\right) B$$

So need to show $\rho \tau + (\rho' \lambda)' = 0$.

Since we have $\rho^2 + (\rho' \lambda)^2 = \text{const}$, differentiating, we have

$$0 = 2\rho\rho' + 2(\rho' \lambda)(\rho' \lambda)'$$

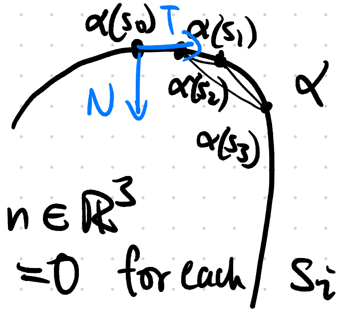
$$= \frac{2\rho'}{\tau}\left(\rho \tau + (\rho' \lambda)'\right).$$

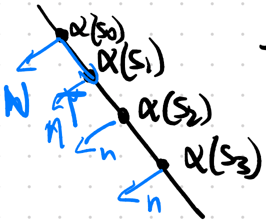
Since $\rho' \neq 0$, $\tau \neq 0$, we must conclude $\rho \tau + (\rho' \lambda)' = 0$.

So $\beta' = 0 \Rightarrow \beta$ is constant, say $\beta(s) \equiv A$. Then

$|\alpha(s) - A|^2 = \rho^2 + (\rho' \lambda)^2 = \text{const}$! Hence, α lies on a sphere

4) As in the lecture notes suppose $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are collinear. Then





there are two fixed vectors $v, n \in \mathbb{R}^3$ such that $\langle \alpha(s_i) - v, n \rangle = 0$ for each s_i.

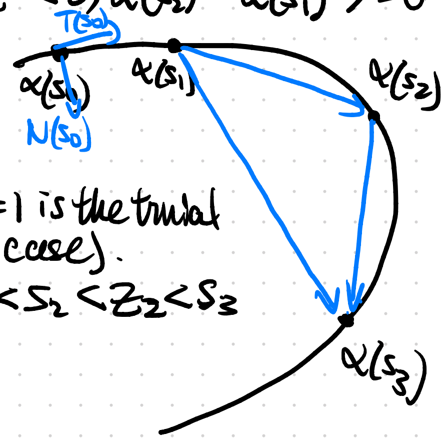
Regarding this as a function $f(s) = \langle \alpha(s) - v, n \rangle$, by MVT we have that there are z_1, z_2 such that $s_1 < z_1 < s_2 < z_2 < s_3$ with $f'(z_1) = f'(z_2) = 0$.

Then applying MVT again there is a η with $z_1 < \eta < z_2$ such that $f''(\eta) = 0$, that is, $\langle \alpha''(\eta), n \rangle = 0$

But as $s_1, s_2, s_3 \to s_0$, $n \to N(s_0)$, and $\alpha''(\eta) = \kappa(s_0) N(s_0)$. which would imply $\kappa = 0$, a contradiction.

2nd Part of the question:

Since $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ are not collinear (for s_1, s_2, s_3 sufficiently close to s_0), they determine a plane with normal vector (up to a sign) b satisfying $\langle b, \alpha(s_3) - \alpha(s_1) \rangle = 0$, $\langle b, \alpha(s_2) - \alpha(s_1) \rangle = 0$



Writing $g(s) = \langle b, \alpha(s) - \alpha(s_1) \rangle$

Then $g(s_i) = 0$ for $i = 1, 2, 3$. ($i = 1$ is the trivial case).

Then by MVT, $\exists\, z_1, z_2$ with $s_1 < z_1 < s_2 < z_2 < s_3$ such that

$g'(z_1) = g'(z_2) = 0$

That is, $\langle b, \alpha'(z_1)\rangle = \langle b, \alpha'(z_2)\rangle = 0$.

Also, by MVT, there is a η s.t. $z_1 < \eta < z_2$ such that

$g''(\eta) = 0$, that is, $\langle b, \alpha''(\eta)\rangle = 0$.

Taking $s_1, s_2, s_3 \to s_0$, $\alpha'(z_i) \longrightarrow T(s_0)$.

$\alpha''(\eta) \to K(s_0) N(s_0)$

Since $K(s_0) > 0$, this implies

$\langle b, T(s_0)\rangle = 0$, $\langle b, N(s_0)\rangle = 0$.

That is, $b \longrightarrow B(s_0)$ (after choosing a sign).

In other words, the plane determined by $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ converges to the one spanned by $T(s_0), N(s_0)$.

5) This problem is straightforward using the Fundamental Theorem of curves.

let $\beta(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right)$, $c^2 = a^2 + b^2$

be a parametrization of a circular helix. Then we can compute $\kappa_\beta(s)$, $\tau_\beta(s)$.

As in class, we get:

$$\kappa(s) = \frac{a}{c^2}$$

$$\tau(s) = \frac{b}{c^2}$$ using $B' = -\tau N$ sign convention.

So for $a > 0$, $b \neq 0$, $\kappa_\beta > 0$, $\tau_\beta \neq 0$ are constants.

So given regular curve $\alpha(s)$ with $\kappa_\alpha > 0$, $\tau_\alpha \neq 0$ constants, choosing a_α, b_α s.t.

$$\frac{a_\alpha}{a_\alpha^2 + b_\alpha^2} = \kappa_\alpha, \quad \frac{b_\alpha}{a_\alpha^2 + b_\alpha^2} = \tau_\alpha$$

which we can always do since $\tau_\alpha \neq 0$, $\kappa_\alpha > 0$, by uniqueness part of fundamental theorem of curves we can see that α is a circular helix with parametrization as above for chosen a_α, b_α.

Alternatively, by Problem 2 above since $\frac{\kappa}{\tau} = \text{const.}$, we can say α is a cylindrical helix. Then need to show that α projected onto T-N plane is a circle. But this is immediate since we know $\kappa > 0$ is constant.

Second Alternatively: explicitly solve the ODE given by Frénet formulas.